

# An Inexact Potential Reduction Method for Linear Programming

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**Technical Report, July 2016**

## Abstract

A class of interior point methods using inexact directions is analysed. The linear system arising in interior point methods for linear programming is reformulated such that the solution is less sensitive to perturbations in the right-hand side. For the new system an implementable condition is formulated that controls the relative error in the solution. Based on this condition, a feasible and an infeasible potential reduction method are described which retain the convergence and complexity bounds known for exact directions.

## 1 Introduction

The primal-dual interior point method (IPM) is one of the most widely used methods for solving large linear programming problems. The method can be analysed and implemented as a path-following algorithm, in which the iterates follow a central trajectory toward the solution set, or as a potential reduction algorithm, which makes progress by systematically reducing a potential function. Most implementations make use of the path-following concept [3].

This paper analyses a variant of the IPM that works with inexact step directions. Inexact directions occur in implementations which solve the linear equation systems by iterative methods. The analysis given here is closely related to such an implementation and provides criteria to control the level of inexactness in the computation.

The paper introduces two interior point algorithms that work with inexact directions. The first algorithm requires a strictly feasible starting point and keeps all iterates feasible. The second algorithm can start from an infeasible point and achieves feasibility in the limit. Both algorithms are formulated and analysed as potential reduction methods. It is proved that in both cases the *inexact* methods retain the convergence and complexity bounds of the *exact* ones.

The linear program is stated in standard form of a primal-dual pair

$$\text{minimize } \mathbf{c}^T \mathbf{x} \quad \text{subject to } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \quad (1)$$

$$\text{maximize } \mathbf{b}^T \mathbf{y} \quad \text{subject to } A^T \mathbf{y} + \mathbf{z} = \mathbf{c}, \mathbf{z} \geq \mathbf{0}, \quad (2)$$

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in which  $A$  is an  $m \times n$  matrix of full row rank. An IPM generates a sequence of iterates  $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k)$  by taking steps along the Newton direction to the nonlinear system

$$F(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \begin{pmatrix} A\mathbf{x} - \mathbf{b} \\ A^T \mathbf{y} + \mathbf{z} - \mathbf{c} \\ X\mathbf{z} - \mu \mathbf{e} \end{pmatrix} = \mathbf{0}, \quad (3)$$

in which  $X := \text{diag}(\mathbf{x})$ ,  $\mathbf{e}$  is the  $n$ -vector of ones and  $\mu > 0$  is a parameter that is gradually reduced to zero. The step directions are computed from the linear system

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z^k & 0 & X^k \end{bmatrix} \begin{pmatrix} \Delta \mathbf{x}^* \\ \Delta \mathbf{y}^* \\ \Delta \mathbf{z}^* \end{pmatrix} = \begin{pmatrix} \mathbf{b} - A\mathbf{x}^k \\ \mathbf{c} - A^T \mathbf{y}^k - \mathbf{z}^k \\ -X^k \mathbf{z}^k + \mu \mathbf{e} \end{pmatrix}, \quad (4)$$

in which  $X^k := \text{diag}(\mathbf{x}^k)$  and  $Z^k := \text{diag}(\mathbf{z}^k)$ . The step sizes are chosen to keep  $\mathbf{x}^k$  and  $\mathbf{z}^k$  positive.

The potential reduction method is a particular instance of the IPM. It sets  $\mu = (\mathbf{x}^k)^T \mathbf{z}^k / (n + \nu)$  for a constant  $\nu \geq \sqrt{n}$  and chooses a step size to decrease a potential function by at least a certain constant. This paper uses the Tanabe-Todd-Ye potential function [9, 10]

$$\phi(\mathbf{x}, \mathbf{z}) := (n + \nu) \ln(\mathbf{x}^T \mathbf{z}) - \sum_{i=1}^n \ln(x_i z_i) - n \ln n.$$

The inexact methods work with step directions of the form

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z^k & 0 & X^k \end{bmatrix} \begin{pmatrix} \Delta \mathbf{x} \\ \Delta \mathbf{y} \\ \Delta \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{b} - A\mathbf{x}^k \\ \mathbf{c} - A^T \mathbf{y}^k - \mathbf{z}^k \\ -X^k \mathbf{z}^k + \mu \mathbf{e} + \xi_0 \end{pmatrix}, \quad (5)$$

in which a residual  $\xi_0$  remains in the complementarity equations. The primal and dual feasibility equations must be satisfied exactly. Conditions will be imposed on  $\xi_0$  to guarantee that the step decreases  $\phi$  sufficiently.

## 2 The Inexact Potential Reduction Method

Considering one iterate  $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k)$ , we define diagonal matrices

$$D := (X^k)^{1/2} (Z^k)^{-1/2}, \quad W := (X^k Z^k)^{1/2},$$

and  $\mathbf{w} := W\mathbf{e}$ . To analyse the step directions it is convenient to write the Newton system (4) in the scaled quantities  $\Delta \mathbf{u}^* := D^{-1} \Delta \mathbf{x}^*$  and  $\Delta \mathbf{v}^* := D \Delta \mathbf{z}^*$ , which is

$$\begin{bmatrix} AD & 0 & 0 \\ 0 & DA^T & I \\ I & 0 & I \end{bmatrix} \begin{pmatrix} \Delta \mathbf{u}^* \\ \Delta \mathbf{y}^* \\ \Delta \mathbf{v}^* \end{pmatrix} = \begin{pmatrix} \mathbf{b} - A\mathbf{x}^k \\ D(\mathbf{c} - A^T \mathbf{y}^k - \mathbf{z}^k) \\ -\mathbf{w} + \mu W^{-1} \mathbf{e} \end{pmatrix} =: \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} \end{pmatrix}. \quad (6)$$

The inexact solution corresponding to a residual  $\xi$  in the scaled system then satisfies

$$\begin{bmatrix} AD & 0 & 0 \\ 0 & DA^T & I \\ I & 0 & I \end{bmatrix} \begin{pmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{y} \\ \Delta \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} + \xi \end{pmatrix}. \quad (7)$$

The inexact potential reduction algorithms make use of the following conditions on the residual, in which  $\kappa \in [0, 1)$  and  $\|\cdot\|$  is the Euclidean norm:

$$-\mathbf{r}^T \boldsymbol{\xi} \leq \kappa \|\mathbf{r}\|^2, \quad (8a)$$

$$\|\boldsymbol{\xi}\| \leq \kappa \min(\|\Delta \mathbf{u}\|, \|\Delta \mathbf{v}\|), \quad (8b)$$

$$-\mathbf{w}^T \boldsymbol{\xi} \leq \kappa n / (n + \nu) \|\mathbf{w}\|^2. \quad (8c)$$

Algorithm 1 is the inexact version of the feasible potential reduction method described in [4, 11]. All iterates belong to the strictly feasible set

$$\mathcal{F}^o := \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) : A\mathbf{x} = \mathbf{b}, A^T \mathbf{y} + \mathbf{z} = \mathbf{c}, (\mathbf{x}, \mathbf{z}) > 0\},$$

which is assumed to be nonempty. The algorithm does not require condition (8c).

**Algorithm 1.** Given  $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0) \in \mathcal{F}^o$  and  $\varepsilon > 0$ . Choose  $\nu \geq \sqrt{n}$  and  $\kappa \in [0, 1)$ . Set  $\delta := 0.15(1 - \kappa)^4$  and  $k := 0$ .

1. If  $(\mathbf{x}^k)^T \mathbf{z}^k \leq \varepsilon$  then stop.
2. Let  $\mu := (\mathbf{x}^k)^T \mathbf{z}^k / (n + \nu)$ . Compute the solution to (7) with residual  $\boldsymbol{\xi}$  that satisfies (8a)–(8b). Set  $\Delta \mathbf{x} := D \Delta \mathbf{u}$  and  $\Delta \mathbf{z} := D^{-1} \Delta \mathbf{v}$ .
3. Find step size  $\alpha^k$  such that

$$\phi(\mathbf{x}^k + \alpha^k \Delta \mathbf{x}, \mathbf{z}^k + \alpha^k \Delta \mathbf{z}) \leq \phi(\mathbf{x}^k, \mathbf{z}^k) - \delta. \quad (9)$$

4. Set  $(\mathbf{x}^{k+1}, \mathbf{y}^{k+1}, \mathbf{z}^{k+1}) := (\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k) + \alpha^k (\Delta \mathbf{x}, \Delta \mathbf{y}, \Delta \mathbf{z})$ ,  $k := k + 1$  and go to 1.

The following theorem, which is proved in Section 3, states that Algorithm 1 retains the complexity bound of the exact version analysed in [4, 11].

**Theorem 1.** Let  $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0) \in \mathcal{F}^o$  and  $L \geq 0$  such that  $\phi(\mathbf{x}^0, \mathbf{z}^0) = O(\nu L)$ . Suppose that  $\ln(1/\varepsilon) = O(L)$ . Then Algorithm 1 terminates in  $O(\nu L)$  iterations provided that  $\kappa$  is chosen independently of  $n$ .

Algorithm 2 is an infeasible inexact potential reduction method, as its sequence of iterates does not, in general, belong to  $\mathcal{F}^o$ . It extends Algorithm 1 from [6] to work with inexact directions. Given positive constants  $\rho$  and  $\varepsilon$ , it finds  $\varepsilon$ -accurate approximations to solutions  $\mathbf{x}^*$  to (1) and  $(\mathbf{y}^*, \mathbf{z}^*)$  to (2), if they exist, such that

$$\|(\mathbf{x}^*, \mathbf{z}^*)\|_\infty \leq \rho.$$

**Algorithm 2.** Given  $\rho > 0$  and  $\varepsilon > 0$ . Set  $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0) = \rho(\mathbf{e}, \mathbf{0}, \mathbf{e})$ . Choose  $\sqrt{n} \leq \nu \leq 2n$  and  $\kappa \in [0, 1)$ . Set  $\delta := (1 - \kappa)^4 / (1600(n + \nu)^2)$  and  $k := 0$ .

1. If  $(\mathbf{x}^k)^T \mathbf{z}^k \leq \varepsilon$  then stop.
2. Let  $\mu := (\mathbf{x}^k)^T \mathbf{z}^k / (n + \nu)$ . Compute the solution to (7) with residual  $\boldsymbol{\xi}$  that satisfies (8a)–(8c). Set  $\Delta \mathbf{x} := D \Delta \mathbf{u}$  and  $\Delta \mathbf{z} := D^{-1} \Delta \mathbf{v}$ .
3. Find step size  $\alpha^k$  such that

$$\phi(\mathbf{x}^k + \alpha^k \Delta \mathbf{x}, \mathbf{z}^k + \alpha^k \Delta \mathbf{z}) \leq \phi(\mathbf{x}^k, \mathbf{z}^k) - \delta, \quad (10a)$$

$$(\mathbf{x}^k + \alpha^k \Delta \mathbf{x})^T (\mathbf{z}^k + \alpha^k \Delta \mathbf{z}) \geq (1 - \alpha^k) (\mathbf{x}^k)^T \mathbf{z}^k. \quad (10b)$$

If no such step size exists then stop.

4. Set  $(\mathbf{x}^{k+1}, \mathbf{y}^{k+1}, \mathbf{z}^{k+1}) := (\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k) + \alpha^k (\Delta \mathbf{x}, \Delta \mathbf{y}, \Delta \mathbf{z})$ ,  $k := k + 1$  and go to 1.

The following theorem, which is proved in Section 4, states that Algorithm 2 retains the complexity bound of the exact infeasible potential reduction method [6].

**Theorem 2.** *Let  $L \geq \ln n$  such that  $\rho = O(L)$ . Suppose that  $\ln(1/\varepsilon) = O(L)$ . Then Algorithm 2 terminates in  $O(\nu(n + \nu)^2 L)$  iterations provided that  $\kappa$  is chosen independently of  $n$ . If the algorithm stops in step 1 then the iterate is an  $\varepsilon$ -approximate solution; otherwise it stops in step 3 showing that there are no optimal solutions  $\mathbf{x}^*$  to (1) and  $(\mathbf{y}^*, \mathbf{z}^*)$  to (2) such that  $\|(\mathbf{x}^*, \mathbf{z}^*)\|_\infty \leq \rho$ .*

The following lemma is key to the analysis of the inexact potential reduction methods given in the next two sections. It exploits the particular form of the scaled Newton system to prove that condition (8b) bounds the *relative error* in the inexact solution.

**Lemma 1.** *Given solutions to (6) and (7), suppose that (8b) holds for  $\kappa \in [0, 1)$ . Then*

$$\frac{\|\Delta \mathbf{u} - \Delta \mathbf{u}^*\|}{\|\Delta \mathbf{u}^*\|} \leq \frac{\kappa}{1 - \kappa}, \quad \frac{\|\Delta \mathbf{v} - \Delta \mathbf{v}^*\|}{\|\Delta \mathbf{v}^*\|} \leq \frac{\kappa}{1 - \kappa}.$$

*Proof.* Denoting  $P := DA^T(AD^2A^T)^{-1}AD$ , the solution to (6) is

$$\begin{aligned} \Delta \mathbf{u}^* &= DA^T(AD^2A^T)^{-1}\mathbf{p} - (I - P)\mathbf{q} + (I - P)\mathbf{r}, \\ \Delta \mathbf{y}^* &= (AD^2A^T)^{-1}(\mathbf{p} + AD\mathbf{q} - AD\mathbf{r}), \\ \Delta \mathbf{v}^* &= -DA^T(AD^2A^T)^{-1}\mathbf{p} + (I - P)\mathbf{q} + P\mathbf{r}. \end{aligned}$$

It follows that

$$\begin{aligned} \Delta \mathbf{u} - \Delta \mathbf{u}^* &= (I - P)\boldsymbol{\xi}, \\ \Delta \mathbf{v} - \Delta \mathbf{v}^* &= P\boldsymbol{\xi}. \end{aligned}$$

Because  $P$  and  $(I - P)$  are projection operators,  $\|P\| \leq 1$  and  $\|(I - P)\| \leq 1$ . Therefore the absolute errors are bounded by the norm of the residual,

$$\begin{aligned} \|\Delta \mathbf{u} - \Delta \mathbf{u}^*\| &\leq \|\boldsymbol{\xi}\|, \\ \|\Delta \mathbf{v} - \Delta \mathbf{v}^*\| &\leq \|\boldsymbol{\xi}\|. \end{aligned}$$

On the other hand, it follows from the triangle inequality and (8b) that

$$\|\Delta \mathbf{u}^*\| = \|\Delta \mathbf{u} - (I - P)\boldsymbol{\xi}\| \geq \|\Delta \mathbf{u}\| - \|\boldsymbol{\xi}\| \geq (1 - \kappa) \|\Delta \mathbf{u}\|, \quad (12a)$$

$$\|\Delta \mathbf{v}^*\| = \|\Delta \mathbf{v} - P\boldsymbol{\xi}\| \geq \|\Delta \mathbf{v}\| - \|\boldsymbol{\xi}\| \geq (1 - \kappa) \|\Delta \mathbf{v}\|. \quad (12b)$$

Combining both inequalities and (8b) gives

$$\begin{aligned} \frac{\|\Delta \mathbf{u} - \Delta \mathbf{u}^*\|}{\|\Delta \mathbf{u}^*\|} &\leq \frac{\|\boldsymbol{\xi}\|}{\|\Delta \mathbf{u}^*\|} \leq \frac{\kappa \|\Delta \mathbf{u}\|}{(1 - \kappa) \|\Delta \mathbf{u}\|} = \frac{\kappa}{1 - \kappa}, \\ \frac{\|\Delta \mathbf{v} - \Delta \mathbf{v}^*\|}{\|\Delta \mathbf{v}^*\|} &\leq \frac{\|\boldsymbol{\xi}\|}{\|\Delta \mathbf{v}^*\|} \leq \frac{\kappa \|\Delta \mathbf{v}\|}{(1 - \kappa) \|\Delta \mathbf{v}\|} = \frac{\kappa}{1 - \kappa} \end{aligned}$$

as claimed.  $\square$

### 3 Proof of Theorem 1

This and the next section use two technical results from Mizuno, Kojima and Todd [6], which are stated in the following two lemmas.

**Lemma 2.** *For any  $n$ -vectors  $\mathbf{x} > 0$ ,  $\mathbf{z} > 0$ ,  $\Delta\mathbf{x}$ ,  $\Delta\mathbf{z}$  and  $\alpha > 0$  such that  $\|\alpha X^{-1}\Delta\mathbf{x}\|_\infty \leq \tau$  and  $\|\alpha Z^{-1}\Delta\mathbf{z}\|_\infty \leq \tau$  for a constant  $\tau \in (0, 1)$  it holds true that*

$$\phi(\mathbf{x} + \alpha\Delta\mathbf{x}, \mathbf{z} + \alpha\Delta\mathbf{z}) \leq \phi(\mathbf{x}, \mathbf{z}) + g_1\alpha + g_2\alpha^2$$

with coefficients

$$g_1 = \left( \frac{n+\nu}{\mathbf{x}^T \mathbf{z}} \mathbf{e} - (XZ)^{-1} \mathbf{e} \right)^T (Z\Delta\mathbf{x} + X\Delta\mathbf{z}),$$

$$g_2 = (n+\nu) \frac{\Delta\mathbf{x}^T \Delta\mathbf{z}}{\mathbf{x}^T \mathbf{z}} + \frac{\|X^{-1}\Delta\mathbf{x}\|^2 + \|Z^{-1}\Delta\mathbf{z}\|^2}{2(1-\tau)}.$$

**Lemma 3.** *For any  $n$ -vector  $\mathbf{w} > 0$  and  $\nu \geq \sqrt{n}$*

$$\left\| W^{-1} \mathbf{e} - \frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \mathbf{w} \right\| \geq \frac{\sqrt{3}}{2w_{\min}},$$

where  $W := \text{diag}(\mathbf{w})$  and  $w_{\min} := \min_i w_i$ .

Applying Lemma 3 to the vector  $\mathbf{r}$  defined in (6) shows that

$$\|\mathbf{r}\| = \left\| -\mathbf{w} + \mu W^{-1} \mathbf{e} \right\| = \mu \left\| -\frac{1}{\mu} \mathbf{w} + W^{-1} \mathbf{e} \right\| \geq \mu \frac{\sqrt{3}}{2w_{\min}}. \quad (13)$$

The following lemma extends the analysis of the feasible potential reduction method given in [11]. It shows that Algorithm 1 finds a step size that reduces  $\phi$  by at least the prescribed value in each iteration.

**Lemma 4.** *In the  $k$ -th iteration of Algorithm 1 (9) holds for*

$$\alpha := \frac{w_{\min}}{2\|\mathbf{r}\|} (1 - \kappa)^3,$$

where  $w_{\min} := \min_i \sqrt{x_i^k z_i^k}$ .

*Proof.* It follows from the first two block equations in (7) and  $\mathbf{p} = \mathbf{0}$ ,  $\mathbf{q} = \mathbf{0}$  that

$$\Delta\mathbf{u}^T \Delta\mathbf{v} = -\Delta\mathbf{u}^T D A^T \Delta\mathbf{y} = -(AD\Delta\mathbf{u})^T \Delta\mathbf{y} = 0,$$

and analogously  $(\Delta\mathbf{u}^*)^T \Delta\mathbf{v}^* = 0$  from (6). Therefore  $\|\Delta\mathbf{u}^*\|^2 + \|\Delta\mathbf{v}^*\|^2 = \|\mathbf{r}\|^2$  and from (12a), (12b) and the definition of  $\alpha$

$$\|\alpha X^{-1}\Delta\mathbf{x}\|_\infty \leq \alpha \|W^{-1}\| \|\Delta\mathbf{u}\| \leq \frac{\alpha}{w_{\min}} \frac{\|\Delta\mathbf{u}^*\|}{1-\kappa} \leq \frac{\alpha}{w_{\min}} \frac{\|\mathbf{r}\|}{1-\kappa} \leq \frac{1}{2},$$

$$\|\alpha Z^{-1}\Delta\mathbf{z}\|_\infty \leq \alpha \|W^{-1}\| \|\Delta\mathbf{v}\| \leq \frac{\alpha}{w_{\min}} \frac{\|\Delta\mathbf{v}^*\|}{1-\kappa} \leq \frac{\alpha}{w_{\min}} \frac{\|\mathbf{r}\|}{1-\kappa} \leq \frac{1}{2}.$$

Therefore  $\tau := 1/2$  satisfies the assumptions of Lemma 2, so that

$$\phi(\mathbf{x}^k + \alpha\Delta\mathbf{x}, \mathbf{z}^k + \alpha\Delta\mathbf{z}) - \phi(\mathbf{x}^k, \mathbf{z}^k) \leq g_1\alpha + g_2\alpha^2$$

with coefficients

$$\begin{aligned} g_1 &= \left( \frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \mathbf{e} - W^{-2} \mathbf{e} \right)^T W(\Delta \mathbf{u} + \Delta \mathbf{v}) \\ g_2 &= \|W^{-1} \Delta \mathbf{u}\|^2 + \|W^{-1} \Delta \mathbf{v}\|^2. \end{aligned}$$

To show that  $\phi$  is sufficiently reduced along the direction  $(\Delta \mathbf{x}, \Delta \mathbf{z})$  it is necessary to show that  $g_1$  is negative and bounded away from zero, while  $g_2$  is bounded. From the definition of  $\mathbf{r}$  and condition (8a) it follows that

$$g_1 = \left( \frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \mathbf{w} - W^{-1} \mathbf{e} \right)^T (\Delta \mathbf{u} + \Delta \mathbf{v}) = -\frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \mathbf{r}^T (\mathbf{r} + \boldsymbol{\xi}) \quad (14a)$$

$$\leq -(1-\kappa) \frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \|\mathbf{r}\|^2. \quad (14b)$$

For the second order term it follows from (12a), (12b) that

$$\begin{aligned} g_2 &= \|W^{-1} \Delta \mathbf{u}\|^2 + \|W^{-1} \Delta \mathbf{v}\|^2 \leq \frac{1}{w_{\min}^2} (\|\Delta \mathbf{u}\|^2 + \|\Delta \mathbf{v}\|^2) \\ &\leq \frac{\|\Delta \mathbf{u}^*\|^2 + \|\Delta \mathbf{v}^*\|^2}{w_{\min}^2 (1-\kappa)^2} = \frac{\|\mathbf{r}\|^2}{w_{\min}^2 (1-\kappa)^2}. \end{aligned}$$

Inserting the bounds on  $g_1$  and  $g_2$  into the quadratic form and using the definition of  $\alpha$  gives

$$\begin{aligned} &\phi(\mathbf{x}^k + \alpha \Delta \mathbf{x}, \mathbf{z}^k + \alpha \Delta \mathbf{z}) - \phi(\mathbf{x}^k, \mathbf{z}^k) \\ &\leq -(1-\kappa) \frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \|\mathbf{r}\|^2 \alpha + \frac{\|\mathbf{r}\|^2}{w_{\min}^2 (1-\kappa)^2} \alpha^2 \\ &= -(1-\kappa)^4 \frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \frac{w_{\min}}{2} \|\mathbf{r}\| + \frac{(1-\kappa)^4}{4}. \end{aligned}$$

Finally, using the bound on  $\|\mathbf{r}\|$  from (13) gives

$$\begin{aligned} &\phi(\mathbf{x}^k + \alpha \Delta \mathbf{x}, \mathbf{z}^k + \alpha \Delta \mathbf{z}) - \phi(\mathbf{x}^k, \mathbf{z}^k) \\ &\leq (1-\kappa)^4 \left( -\frac{\sqrt{3}}{4} + \frac{1}{4} \right) \\ &\leq -0.15(1-\kappa)^4 = -\delta \end{aligned}$$

as claimed.  $\square$

The proof of Theorem 1 is immediate. Since  $\phi(\mathbf{x}, \mathbf{z}) \geq \nu \ln(\mathbf{x}^T \mathbf{z})$ , the termination condition

$$\nu \ln((\mathbf{x}^k)^T \mathbf{z}^k) \leq \nu \ln(\varepsilon)$$

is satisfied when

$$\phi(\mathbf{x}^k, \mathbf{z}^k) \leq \phi(\mathbf{x}^0, \mathbf{z}^0) - k\delta \leq \nu \ln(\varepsilon). \quad (15)$$

Since under the assumption of the theorem  $\phi(\mathbf{x}^0, \mathbf{z}^0) = O(\nu L)$  and  $\ln(1/\varepsilon) = O(L)$ , and since  $\delta$  is independent of  $n$ , (15) holds for  $k \geq K = O(\nu L)$ .

## 4 Proof of Theorem 2

The proof of the theorem is based on Mizuno, Kojima and Todd [6]. We define a sequence  $\{\theta^k\}$  by

$$\theta^0 := 1 \quad \text{and} \quad \theta^{k+1} := (1 - \alpha^k)\theta^k \text{ for } k \geq 0. \quad (16)$$

Since the first two block equations in (3) are linear and satisfied exactly by a full step of the algorithm

$$(A\mathbf{x}^k - \mathbf{b}, A^T\mathbf{y}^k + \mathbf{z}^k - \mathbf{c}) = \theta^k (A\mathbf{x}^0 - \mathbf{b}, A^T\mathbf{y}^0 + \mathbf{z}^0 - \mathbf{c}).$$

The following lemma is obtained from Lemma 4 in [6] by setting  $\gamma_0 = 1$  and  $\gamma_1 = 1$ .

**Lemma 5.** *Let  $\rho > 0$  and suppose that*

$$\begin{aligned} (\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0) &= \rho(e, \mathbf{0}, e), \\ (A\mathbf{x}^k - \mathbf{b}, A^T\mathbf{y}^k + \mathbf{z}^k - \mathbf{c}) &= \theta^k (A\mathbf{x}^0 - \mathbf{b}, A^T\mathbf{y}^0 + \mathbf{z}^0 - \mathbf{c}), \\ (\mathbf{x}^k)^T \mathbf{z}^k &\geq \theta^k (\mathbf{x}^0)^T \mathbf{z}^0. \end{aligned} \quad (17)$$

*If there exist solutions  $\mathbf{x}^*$  to (1) and  $(\mathbf{y}^*, \mathbf{z}^*)$  to (2) such that  $\|(\mathbf{x}^*, \mathbf{z}^*)\|_\infty \leq \rho$  then the solution to (6) at  $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k)$  satisfies*

$$\begin{aligned} \|\Delta \mathbf{u}^*\| &\leq \frac{5(\mathbf{x}^k)^T \mathbf{z}^k}{w_{\min}}, \\ \|\Delta \mathbf{v}^*\| &\leq \frac{5(\mathbf{x}^k)^T \mathbf{z}^k}{w_{\min}}, \end{aligned}$$

where  $w_{\min} := \min_i \sqrt{x_i^k z_i^k}$ .

The following lemma is based on Lemma 5 in [6]. It shows that when optimal solutions to (1) and (2) exist, then Algorithm 2 can find a step size in each iteration that satisfies (10a) and (10b).

**Lemma 6.** *If there exist optimal solutions  $\mathbf{x}^*$  to (1) and  $(\mathbf{y}^*, \mathbf{z}^*)$  to (2) such that  $\|(\mathbf{x}^*, \mathbf{z}^*)\|_\infty \leq \rho$  then (10a) and (10b) hold for*

$$\alpha := \frac{(1 - \kappa)^3 w_{\min}^2}{200(n + \nu)(\mathbf{x}^k)^T \mathbf{z}^k}$$

*in the  $k$ -th iteration, where  $w_{\min} := \min_i \sqrt{x_i^k z_i^k}$ .*

*Proof.* A simple calculation shows that by definition of  $(\mathbf{x}^0, \mathbf{z}^0)$  and because of (10b) the assumptions of Lemma 5 are satisfied. Combining the lemma with (12a), (12b) shows that

$$\begin{aligned} \|\Delta \mathbf{u}\| &\leq \frac{5(\mathbf{x}^k)^T \mathbf{z}^k}{(1 - \kappa)w_{\min}}, \\ \|\Delta \mathbf{v}\| &\leq \frac{5(\mathbf{x}^k)^T \mathbf{z}^k}{(1 - \kappa)w_{\min}}. \end{aligned}$$

It follows that

$$\begin{aligned} \|\alpha X^{-1} \Delta \mathbf{x}\| &\leq \alpha \|W^{-1}\| \|\Delta \mathbf{u}\| \leq \alpha \frac{5(\mathbf{x}^k)^T \mathbf{z}^k}{(1 - \kappa)w_{\min}^2} = \frac{(1 - \kappa)^2}{40(n + \nu)} \leq \frac{1}{40}, \\ \|\alpha Z^{-1} \Delta \mathbf{z}\| &\leq \alpha \|W^{-1}\| \|\Delta \mathbf{v}\| \leq \alpha \frac{5(\mathbf{x}^k)^T \mathbf{z}^k}{(1 - \kappa)w_{\min}^2} = \frac{(1 - \kappa)^2}{40(n + \nu)} \leq \frac{1}{40}. \end{aligned}$$

Therefore  $\tau := 1/40$  satisfies the assumption of Lemma 2, so that

$$\phi(\mathbf{x}^k + \alpha \Delta \mathbf{x}, \mathbf{z}^k + \alpha \Delta \mathbf{z}) \leq \phi(\mathbf{x}^k, \mathbf{z}^k) + g_1 \alpha + g_2 \alpha^2$$

with coefficients

$$\begin{aligned} g_1 &= \left( \frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \mathbf{e} - W^{-2} \mathbf{e} \right)^T W(\Delta \mathbf{u} + \Delta \mathbf{v}), \\ g_2 &= \left( (n+\nu) \frac{\Delta \mathbf{u}^T \Delta \mathbf{v}}{\mathbf{w}^T \mathbf{w}} + \frac{\|W^{-1} \Delta \mathbf{u}\|^2 + \|W^{-1} \Delta \mathbf{v}\|^2}{2(1-\tau)} \right). \end{aligned}$$

It will be shown that  $g_1$  is negative and bounded away from zero, while  $g_2$  is bounded. Combining (14b) and (13) gives

$$g_1 \leq -(1-\kappa) \frac{1}{\mu} \|\mathbf{r}\|^2 \leq -(1-\kappa) \mu \frac{3}{4w_{\min}^2}.$$

Next, from the bound on  $\|\Delta \mathbf{u}\|$  and  $\|\Delta \mathbf{v}\|$  it follows that

$$|\Delta \mathbf{u}^T \Delta \mathbf{v}| \leq \|\Delta \mathbf{u}\| \|\Delta \mathbf{v}\| \leq \left( \frac{5\mathbf{w}^T \mathbf{w}}{(1-\kappa)w_{\min}} \right)^2, \quad (18)$$

which implies that

$$(n+\nu) \frac{\Delta \mathbf{u}^T \Delta \mathbf{v}}{\mathbf{w}^T \mathbf{w}} \leq \frac{n+\nu}{\mathbf{w}^T \mathbf{w}} \left( \frac{5\mathbf{w}^T \mathbf{w}}{(1-\kappa)w_{\min}} \right)^2 \leq \frac{n+\nu}{n} \left( \frac{5\mathbf{w}^T \mathbf{w}}{(1-\kappa)w_{\min}^2} \right)^2, \quad (19)$$

where the last inequality is obtained by multiplying with  $\mathbf{w}^T \mathbf{w} / (nw_{\min}^2) \geq 1$ . Moreover, the bound on  $\|\Delta \mathbf{u}\|$  and  $\|\Delta \mathbf{v}\|$  also implies that

$$\frac{\|W^{-1} \Delta \mathbf{u}\|^2 + \|W^{-1} \Delta \mathbf{v}\|^2}{2(1-\tau)} \leq \frac{1}{1-\tau} \left( \frac{5\mathbf{w}^T \mathbf{w}}{(1-\kappa)w_{\min}^2} \right)^2. \quad (20)$$

Adding up (19) and (20) and using  $\nu \leq 2n$  gives

$$g_2 \leq \left( \frac{n+\nu}{n} + \frac{1}{1-\tau} \right) \left( \frac{5\mathbf{w}^T \mathbf{w}}{(1-\kappa)w_{\min}^2} \right)^2 \leq 5 \left( \frac{5\mathbf{w}^T \mathbf{w}}{(1-\kappa)w_{\min}^2} \right)^2.$$

Inserting  $g_1, g_2$  and the definition of  $\alpha$  into the quadratic form gives

$$\begin{aligned} &\phi(\mathbf{x}^k + \alpha \Delta \mathbf{x}, \mathbf{z}^k + \alpha \Delta \mathbf{z}) - \phi(\mathbf{x}^k, \mathbf{z}^k) \\ &\leq -(1-\kappa) \frac{\mathbf{w}^T \mathbf{w}}{n+\nu} \frac{3}{4w_{\min}^2} \alpha + 5 \left( \frac{5\mathbf{w}^T \mathbf{w}}{(1-\kappa)w_{\min}^2} \right)^2 \alpha^2 \\ &= \frac{(1-\kappa)^4}{(n+\nu)^2} \left( -\frac{3}{4 \cdot 200} + 5 \left( \frac{5}{200} \right)^2 \right) = -\delta, \end{aligned}$$

which shows that  $\alpha$  satisfies (10a).

Finally, to verify that  $\alpha$  satisfies (10b), a straightforward calculation shows that

$$\Delta \mathbf{z}^T \mathbf{x}^k + \Delta \mathbf{x}^T \mathbf{z}^k = \Delta \mathbf{v}^T \mathbf{w} + \Delta \mathbf{u}^T \mathbf{w} = \mathbf{w}^T (\mathbf{r} + \boldsymbol{\xi}) = \left( \frac{n}{n+\nu} - 1 \right) \mathbf{w}^T \mathbf{w} + \mathbf{w}^T \boldsymbol{\xi}$$



and consequently

$$\begin{aligned} (\mathbf{x}^k + \alpha \Delta \mathbf{x})^T (\mathbf{z}^k + \alpha \Delta \mathbf{z}) &= (\mathbf{x}^k)^T \mathbf{z}^k + \alpha (\Delta \mathbf{z}^T \mathbf{x}^k + \Delta \mathbf{x}^T \mathbf{z}^k) + \alpha^2 \Delta \mathbf{x}^T \Delta \mathbf{z} \\ &= (1 - \alpha) \mathbf{w}^T \mathbf{w} + \alpha \left( \frac{n}{n + \nu} \mathbf{w}^T \mathbf{w} + \mathbf{w}^T \boldsymbol{\xi} + \alpha \Delta \mathbf{u}^T \Delta \mathbf{v} \right). \end{aligned}$$

Using (18) and (8c) it follows for the term in parenthesis that

$$\begin{aligned} \frac{n}{n + \nu} \mathbf{w}^T \mathbf{w} + \mathbf{w}^T \boldsymbol{\xi} + \alpha \Delta \mathbf{u}^T \Delta \mathbf{v} &\geq \frac{(1 - \kappa)n}{n + \nu} \mathbf{w}^T \mathbf{w} - \alpha \left( \frac{5 \mathbf{w}^T \mathbf{w}}{(1 - \kappa) w_{\min}} \right)^2 \\ &= \frac{(1 - \kappa) \mathbf{w}^T \mathbf{w}}{n + \nu} \left( n - \frac{1}{8} \right) > 0. \end{aligned}$$

Therefore  $\alpha$  satisfies (10b), which completes the proof.  $\square$

Theorem 2 follows from the lemma by the same argumentation as in [6]. Under the hypothesis of the theorem  $\phi(\mathbf{x}^0, \mathbf{z}^0) = O(\nu L)$  and  $\ln(1/\varepsilon) = O(L)$ . Since  $\phi(\mathbf{x}, \mathbf{z}) \geq \nu \ln(\mathbf{x}^T \mathbf{z})$  and the potential function decreases by at least  $\delta$  in each iteration, Algorithm 2 terminates in  $O(\nu L/\delta) = O(\nu(n + \nu)^2 L)$  iterations. When the algorithm stops in step 1, then  $(\mathbf{x}^k)^T \mathbf{z}^k \leq \varepsilon$  and because of (10b)

$$\|(\mathbf{A}\mathbf{x}^k - \mathbf{b}, \mathbf{A}^T \mathbf{y}^k + \mathbf{z}^k - \mathbf{c})\| \leq \varepsilon \|(\mathbf{A}\mathbf{x}^0 - \mathbf{b}, \mathbf{A}^T \mathbf{y}^0 + \mathbf{z}^0 - \mathbf{c})\| / (\mathbf{x}^0)^T \mathbf{z}^0,$$

so that the final iterate is indeed an  $\varepsilon$ -approximate solution. On the other hand, if there exist optimal solutions  $\mathbf{x}^*$  to (1) and  $(\mathbf{y}^*, \mathbf{z}^*)$  to (2) such that  $\|(\mathbf{x}^*, \mathbf{z}^*)\| \leq \rho$ , then it follows from Lemma 6 that a step size exists which satisfies (10a) and (10b). Therefore, if the algorithm stops in step 3, then there are no such solutions.

**Remark 1.** Theorem 2 imposed the upper bound  $\nu \leq 2n$ , which is not needed in the analysis of the exact potential reduction method. The actual value of this bound, however, is not important and the proof remains valid by adapting  $\alpha$  and  $\delta$  as long as  $\nu = O(n)$ .

## 5 Discussion

The analysis has shown some insights into the conditions (8a)–(8c). It has been seen from (14a) that  $-\mathbf{r}^T \boldsymbol{\xi} < \|\mathbf{r}\|^2$  is sufficient and necessary for  $(\Delta \mathbf{x}, \Delta \mathbf{z})$  to be a descent direction for  $\phi$ , making (8a) a necessary condition in a potential reduction method. Condition (8b) bounds the curvature of  $\phi$  along  $(\Delta \mathbf{x}, \Delta \mathbf{z})$ . When the iterate is feasible this condition can be replaced by  $\|\mathbf{r}\| \leq c \|\boldsymbol{\xi}\|$  for an arbitrary constant  $c$ , since then

$$\|\Delta \mathbf{u}\|^2 + \|\Delta \mathbf{v}\|^2 = \|\mathbf{r} + \boldsymbol{\xi}\|^2 \leq (1 + c)^2 \|\mathbf{r}\|^2$$

gives the required bound on  $g_2$  in Lemma 4. For an infeasible iterate, however, condition (8b) is needed in its form to bound  $\|\Delta \mathbf{u}\|$  and  $\|\Delta \mathbf{v}\|$ . Finally, condition (8c) guarantees that in the infeasible algorithm the step size restriction (10b) can be satisfied.

Inexact directions of the form (5) have been used and analysed in [1, 7] in the path-following method, which sets  $\mu = \sigma \mathbf{x}^T \mathbf{z} / n$  for  $\sigma < 1$  and chooses the step size to keep  $x_i z_i \geq \gamma \mathbf{x}^T \mathbf{z} / n$  for a constant  $\gamma \in (0, 1)$ . Both papers use a basic-nonbasic splitting of the variables and solve (7) with residual  $\boldsymbol{\xi} = (\boldsymbol{\xi}_B, \boldsymbol{\xi}_N) = (\boldsymbol{\xi}_B, \mathbf{0})$ . [7] imposes the condition

$$\|\boldsymbol{\xi}_B\| \leq \frac{(1 - \gamma)\sigma}{4\sqrt{n}} \sqrt{\mathbf{x}^T \mathbf{z} / n}, \quad (21)$$

whereas

$$\|W_B \xi_B\|_\infty \leq \eta x^T z / n \quad (22)$$

is used in [1] with  $\eta < 1$  depending on  $\sigma$  and  $\gamma$ . Both conditions seem to require more effort by an iterative method than the conditions used in this paper. (21) obviously becomes restrictive for large problems. (22) is not affected by the problem dimension, but the infinity norm does not tolerate outliers in  $W_B \xi_B$ .

Another form of inexact direction has been analysed in [5], which solves the complementarity equations exactly and allows a residual in the primal and dual equations. Due to the form of the Newton system, solving the complementarity equations exactly is trivial, whereas computing directions that satisfy primal feasibility requires particular preconditioning techniques [2, 7, 8]. The analysis in [5] shows, however, that a residual in the feasibility equations must be measured in a norm depending on  $A$ , which seems not to be accessible in an implementation. Therefore this form of inexact direction is hardly useful in practice.

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